Motion of vortex ring with tracer particles in superfluid helium

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Recent experiments on quantum turbulence in superfluid helium made use of small tracer particles to track the motion of quantized vortices, determine velocity statistics and visualize vortex reconnections. A problem with this visualization technique is that it may change the turbulent flow which it visualizes. To address this problem, we derive and solve the equations of motion of a quantized vortex ring which contains a given number of particles trapped on the vortex core, hence derive a quantitative criterion to determine in which measure small particles trapped in quantized vortices affect vortex motion. Finally we apply the criterion to a recent experiment.

I. MOTIVATION

The study of quantized vortices[1] in superfluid helium (He II) and of quantum turbulence[2–8] has been held back over the years by the difficulty of flow visualization near absolute zero. Fortunately the problem has been recognized and new visualization techniques have become recently available. One of the most promising technique is based on trapping small micron-size tracer particles (made of glass, polymers, or solid hydrogen) onto quantized vortices.[9,10] The use of tracer particles has made possible, for example, to study velocity statistics[11] in superfluid turbulence and to visualize[12] individual reconnections of quantized vortices, a process which is crucial in the dynamics of turbulence.[13]

To interpret these recent experiments it is necessary to understand the interaction of the tracer particles with the quantized vortices.[14–16] In particular, we need to find in which measure the tracer particles disturb the vortices, hence change the flow which one tries to visualize. Clearly, on one hand a large number of tracers improve the images and the signal to analyze but on the other hand too many tracers must affect the motion of vortices in a significant way. What is the maximum density of tracers which can be used?

To answer this practical question, we introduce a quantitive measure of the tracers' disturbance, which we call the superfluid Stokes number. After deriving and solving the governing equations, we determine by what amount the motion of a quantized vortex ring of radius R is disturbed by the presence of N tracer particles of radius a trapped into the vortex core. By interpreting R as the typical local radius of curvature ℓ of the filaments ($\ell \approx L^{-1/2}$, where L is the vortex line density), we apply the result to a recent experiment in which quantum turbulence was visualized.

II. EQUATIONS OF MOTION

Consider a quantized vortex ring of radius R which moves in the z direction in superfluid helium at temperature T. N small buoyant tracer particles of mass m and radius a are trapped in the vortex core. For the sake of simplicity we assume that the vortex ring remains axisymmetric during the evolution and that the particles remain trapped in it. Let

(r, θ, z) be cylindrical coordinates. The vortex position and velocity are $\mathbf{r}_L = (R, 0, z)$ and $\mathbf{v}_L = (\dot{R}, 0, \dot{z}) = (u, 0, v)$, where $R = R(t)$, $z = z(t)$, t is time, and the dot denotes a time derivative. Let the circulation vector be $\boldsymbol{\kappa} = (0, \kappa, 0)$, where $\kappa = 9.97 \times 10^{-4}$ cm^2/s and let $\hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa}/\kappa$ be the unit vector tangent to the ring in the θ direction. The forces acting on the vortex ring are the Magnus force $\mathbf{F}_M$, the friction force $\mathbf{F}_D$ with the normal fluid, and the Stokes force $\mathbf{F}_S$ induced by the tracer particles. The Magnus force is $\mathbf{F}_M = 2\pi R \mathbf{f}_M$, where

$$\mathbf{f}_M = \rho_s \boldsymbol{\kappa} \times (\mathbf{v}_L - \mathbf{v}_{st}), \tag{1}$$

ρ_s is the superfluid density, $\mathbf{v}_{st} = \mathbf{v}_s + \mathbf{v}_i$ is the total superfluid velocity at the vortex line, and $\mathbf{v}_i = (0, 0, v_i)$ is the self-induced velocity of the ring at $T = 0$, where

$$v_i = \frac{\kappa}{4\pi R}\left[\ln(8R/\xi) - \frac{1}{2}\right], \tag{2}$$

$\xi \approx 10^{-8}$ cm is the vortex core radius, and $\mathbf{v}_s = (0, 0, v_s)$ is an externally applied superflow, which for the sake of simplicity we assume in the z direction. The friction force $\mathbf{F}_D$ arises from the interaction of phonons and rotons (which make up the normal fluid) with the vortex core; we write $\mathbf{F}_D = 2\pi R \mathbf{f}_D$, where the friction per unit length is

$$\mathbf{f}_D = \rho_s \kappa [\Gamma_0 (\mathbf{v}_n - \mathbf{v}_L) + \Gamma_0' \hat{\boldsymbol{\kappa}} \times (\mathbf{v}_n - \mathbf{v}_L)] \tag{3}$$

and $\mathbf{v}_n = (0, 0, v_n)$ is an externally applied normal-fluid velocity, which again we assume to be in the z direction. The quantities Γ_0 and Γ_0' are dimensionless temperature-dependent friction coefficients related to the friction coefficients γ_0 and γ_0' calculated by Barenghi, Donnelly and Vinen[17] by

$$\Gamma_0 = \frac{\gamma_0}{\rho_s \kappa}, \quad \Gamma_0' = \frac{\gamma_0'}{\rho_s \kappa}. \tag{4}$$

Note that Γ_0 and Γ_0' can be expressed in terms of the more used friction coefficients B and B' as[17]

$$\gamma_0 = \frac{\rho_n \rho}{2\rho} \kappa \frac{B}{[1 - B'\rho_n/(2\rho)]^2 + B^2\rho_n^2/(4\rho^2)}, \tag{5}$$

$$\gamma_0' = \frac{\rho_n \rho}{2\rho} \kappa \frac{[B^2 \rho_n/(2\rho) - B'(1 - B' \rho_n/(2\rho))]}{[1 - B' \rho_n/(2\rho)]^2 + B^2 \rho_n^2/(4\rho^2)}. \quad (6)$$

If N spherical particles of mass $m = 4\pi a^3 \rho/3$ each, where ρ_p is the particle density, are attached to the vortex ring, the ring has total mass mN and experiences the Stokes force

$$\mathbf{F}_S = 6\pi a \nu_n \rho_n N(\mathbf{v}_n - \mathbf{v}_L), \quad (7)$$

where $\nu_n = \mu/\rho_n$ is the normal-fluid kinematic viscosity, μ is the viscosity, and ρ_n is the normal-fluid density. This simple linear Stokes drag is appropriate because the Reynolds number based on the particle size is extremely small. For the sake of simplicity and to make contact with some experiments[11,12] we assume that the particles are buoyant, i.e., they have density $\rho_p = \rho = \rho_n + \rho_s$, hence there is no Archimedes force.

The equations of motion of the vortex ring are thus

$$\frac{d\mathbf{r}_L}{dt} = \mathbf{v}_L, \quad (8)$$

$$mN\frac{d\mathbf{v}_L}{dt} = 2\pi R(\mathbf{f}_M + \mathbf{f}_D) + \mathbf{F}_S. \quad (9)$$

The resulting equations for $R(t)$, $z(t)$, $u(t) = \dot{R}$, and $v(t) = \dot{z}$ are

$$\dot{R} = u, \quad (10)$$

$$\dot{z} = v, \quad (11)$$

$$\dot{u} = cR[v - v_s - v_i - \Gamma_0 u + \Gamma_0'(v_n - v)] - \frac{1}{\tau}u, \quad (12)$$

$$\dot{v} = cR[-(1 - \Gamma_0')u + \Gamma_0(v_n - v)] + \frac{1}{\tau}(v_n - v), \quad (13)$$

where

$$c = \frac{2\pi \rho_s \kappa}{mN}, \quad \tau = \frac{m}{6\pi a \nu_n \rho_n}. \quad (14)$$

The quantity τ is the Stokes relaxation time of the particles.

It is convenient to rewrite the equations of motion in dimensionless form. We choose the initial radius $R_0 = R(0)$ as the unit of length and $V_0 = \kappa/R_0$ as the unit of speed; the unit of time is thus R_0^2/κ. We obtain

$$\dot{R} = u, \quad (15)$$

$$\dot{z} = v, \quad (16)$$

$$v - v_s - v_i - \Gamma_0 u + \Gamma_0'(v_n - v) = \frac{1}{R}(\epsilon\dot{u} + \zeta u), \quad (17)$$

$$-(1 - \Gamma_0')u + \Gamma_0(v_n - v) = \frac{1}{R}(\epsilon\dot{v} + \zeta v), \quad (18)$$

where t, R, z, u, v, v_i, v_n, and v_s are now dimensionless. The dimensionless quantities ϵ and ζ are defined as

$$\epsilon = \frac{2N}{3}\left(\frac{a}{R_0}\right)^3\left(\frac{\rho}{\rho_s}\right), \quad \zeta = 3N\left(\frac{R_0}{a}\right)\left(\frac{\rho_n}{\rho_s}\right)\left(\frac{\nu_n}{\kappa}\right). \quad (19)$$

The above dimensionless form of the equations of motion allows us to compare the motion of a vortex ring with N tracers against the motion of a bare vortex ring ($N=0$). If $N=0$ then $\epsilon = \zeta = 0$, and Eqs. (17) and (18) reduce to

$$v - v_s - v_i - \Gamma_0 u + \Gamma_0'(v_n - v) = 0, \quad (20)$$

$$-(1 - \Gamma_0')u + \Gamma_0(v_n - v) = 0. \quad (21)$$

The solution is

$$\dot{R} = u = \frac{\Gamma_0(v_n - v_s - v_i)}{[(1 - \Gamma_0')^2 + \Gamma_0^2]}, \quad (22)$$

$$\dot{z} = v = \frac{(1 - \Gamma_0')(v_s + v_i) + v_n[(1 - \Gamma_0')\Gamma_0' - \Gamma_0^2]}{[(1 - \Gamma_0')^2 + \Gamma_0^2]} \quad (23)$$

as found in Ref. 17. If we further set $v_n = v_s = 0$ we obtain

$$\dot{R} = u = \frac{-\Gamma_0 v_i}{(1 - \Gamma_0')^2 + \Gamma_0^2},$$

$$\dot{z} = v = \frac{(1 - \Gamma_0')v_i}{(1 - \Gamma_0')^2 + \Gamma_0^2}, \quad (24)$$

which means that, as expected, the ring shrinks ($u < 0$) and speeds up, transferring its kinetic energy to the stationary background normal fluid. If we further set $T = 0$ we obtain the expected stationary solutions $u = 0$ and $v = v_i$.

III. RESULTS

We choose values of parameters which are relevant to typical experiments ($T = 2$ K, $a = 10^{-4}$ cm, and $R_0 = 10^{-2}$ cm) and solve Eqs. (15)–(18) numerically using the Adams-Bashforth method. In all calculations we take $v_n = v_s = 0$ for simplicity. The initial conditions are given by Eqs. (22) and (23). During the evolution the vortex ring shrinks. Clearly, when a significant fraction of the circumference of the ring is covered by tracers, our model breaks down, as there is not enough vortex length to provide the correct induced velocity v_i. Since we are not interested in this last stage of the evolution, we arbitrarily stop our calculations when the tracer particles cover one third of the circumference, that is, when the dimensionless ring's radius becomes smaller than $(3N/\pi)(a/R_0)$.

Figure 1 compares the evolution of the dimensionless radius $R(t)$ of vortex rings with $N = 10$, 20, 40, and 80 trapped tracer particles to the decay of a bare vortex ring [$N = 0$, solid curve labeled (a)], computed solving Eqs. (22) and (23). It is apparent that the more particles are trapped (hence the larger ϵ and ζ are), the faster the decay of the ring is compared to the decay of a bare ring.

Figure 2 plots the corresponding time dependence of the radial velocity component $u = \dot{R}$. Initially the ring shrinks at constant rate, then the radius decreases faster and faster.
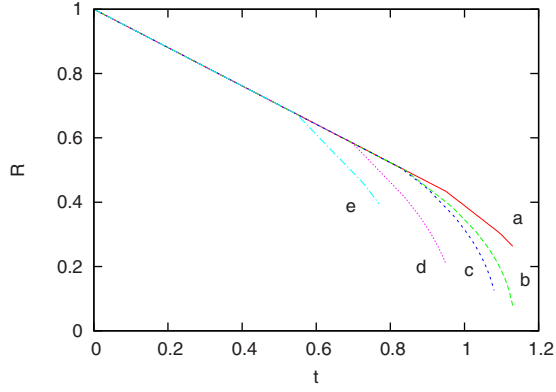
FIG. 1. (Color online) Dimensionless radius R of the vortex ring vs time t for: (a) solid (red) line: bare vortex ring ($N=0$); (b) long-dashed (green) line: vortex ring with $N=10$ tracers, $\epsilon=0.19\times10^{-5}$, and $\zeta=0.34\times10^{-1}$; (c) short-dashed (blue) line: $N=20$, $\epsilon=0.37\times10^{-5}$, and $\zeta=0.68\times10^{-1}$; (d) dotted (magenta) line: $N=40$, $\epsilon=0.75\times10^{-5}$, and $\zeta=0.14$; "e," dashed-dotted (cyan) line: $N=80$, $\epsilon=0.15\times10^{-4}$, and $\zeta=0.27$.

The axial velocity component $\dot{z}=v$ is shown in Fig. 3. Initially v remains similar to the velocity of the bare ring; during the final part of the evolution a vortex ring with tracers moves much faster than a bare vortex ring, because, as shown in Fig. 1, the radius has become much smaller.

The results described above suggest that the evolution of vortex rings which contain tracers is sufficiently similar to the evolution of a bare vortex ring, provided that the ring is not too small and that it does not contain too many tracers. The important question is how to quantify the perturbation which the tracers induce on the ring's evolution so that one can decide whether the visualization with tracers affects or not the evolution of vortices. To answer this question we start by remarking that the ring's decay is caused by both friction and Stokes forces; going back to dimensional variables, we note that the ratio of the magnitudes of these forces is

$$S = \frac{F_S}{F_D} = \frac{3N}{\Gamma_0}\left(\frac{a}{R}\right)\left(\frac{v_n}{\kappa}\right)\left(\frac{\rho_n}{\rho_s}\right) = \frac{\zeta}{\Gamma_0}\left(\frac{R_0}{R}\right). \quad (25)$$

We call S the superfluid Stokes number. In the absence of external forcing ($v_n=v_s=0$) the superfluid Stokes number de-
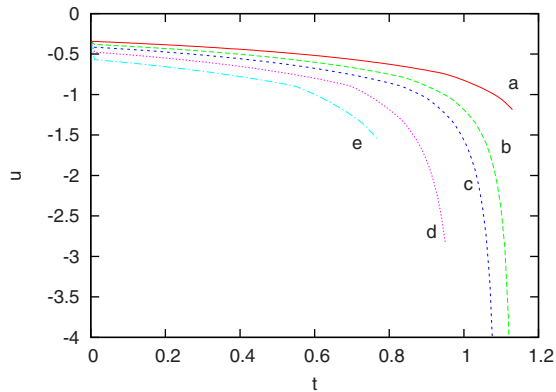

FIG. 2. (Color online) Dimensionless radial velocity u of vortex ring vs time t for $N=0$ [bare ring, solid line labeled (a)] and $N=10, 20, 40,$ and 80 tracers. [Labels and line styles (colors) correspond to those of Fig. 1.]
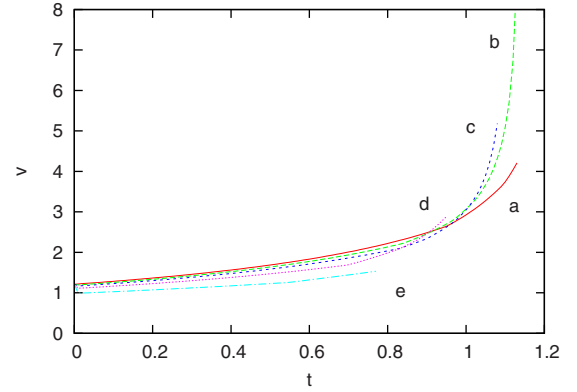
creases during the evolution, starting from the initial value $S(0)=\zeta/\Gamma_0$ at $t=0$ because R decreases with time. Figure 4 shows the time evolution of S corresponding to Figs. 1–3. It is apparent that if $S\ll1$ the evolution of a vortex ring with tracers is very similar to that of a bare vortex ring but if S becomes of order unity or larger then the tracers affect the ring's motion significantly.

To put in evidence the temperature dependence of the superfluid Stokes number, we rewrite Eq. (25) as

$$S = \delta\beta, \quad (26)$$

where

$$\delta = \frac{Na}{2\pi R} \quad (27)$$

is the distance between tracers along the circumference of the ring in units of tracer size, and

$$\beta = \frac{6\pi}{\Gamma_0}\left(\frac{v_n}{\kappa}\right)\left(\frac{\rho_n}{\rho_s}\right) \quad (28)$$

is a strongly temperature-dependent prefactor. Figure 5 shows that β is approximately constant for temperatures be-

FIG. 3. (Color online) Dimensionless axial velocity v of vortex ring vs time t for $N=0$ [bare ring, solid (red) line labeled (a)] and $N=10, 20, 40,$ and 80 tracers. Labels and line styles (colors) correspond to those of Figs. 1 and 2.
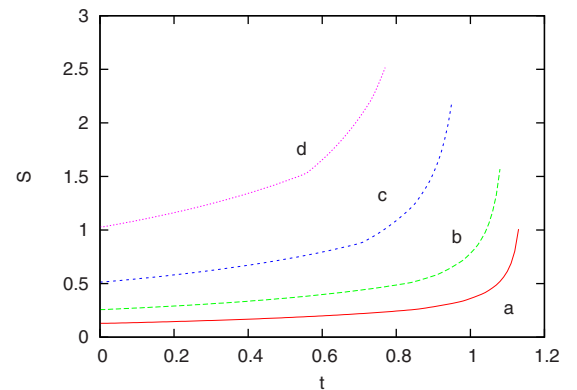

FIG. 4. (Color online) Superfluid Stokes number vs dimensionless time t for $N=10, 20, 40,$ and 80 tracers. Labels and line styles (colors) correspond to those of Figs. 1–3.
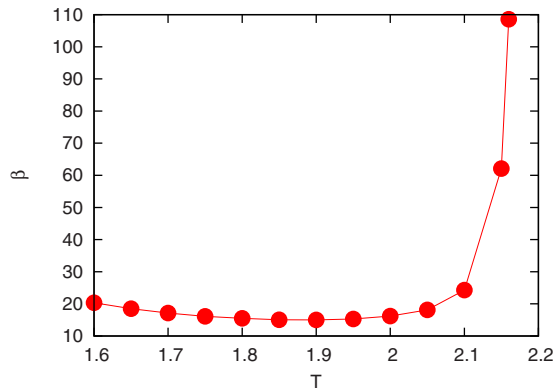
FIG. 5. (Color online) Dimensionless temperature-dependent prefactor β vs temperature T (k).

low 2 K but it increases sharply above 2 K. This means that, for the same geometry (ring's radius and number of tracers along the ring), the superfluid Stokes number is bigger for $T > 2$ K, hence the motion of vortex rings is much more disturbed by the presence of tracer particles than shown for example in Figs. 1–3.

IV. APPLICATION TO QUANTUM TURBULENCE

Isotropic homogeneous quantum turbulence is characterized by the vortex line density $L = \Lambda/V$, where Λ is the total vortex length and V is the volume. In the first approximation this form of turbulence can be generated by applying a heat flux.[18] From the value of L one infers that the average vortex separation and the typical radius of curvature are on the order of $\ell \sim L^{-1/2}$. In a recent experiment,[10] solid hydrogen particles were used to visualize such turbulence: some tracers moved along the normal fluid in the direction of the heat flux and other tracers were trapped on the quantized vortices as the vortex tangle drifted in the opposite direction. At $T = 2$ K the turbulence generated by the heat flux $\dot{Q} = 90$ mW/cm^2 corresponds to approximately[19] $\ell \approx 0.006$ cm. The hydrogen volume fraction used in the experiment was $\phi = V_{H_2}/V \approx 10^{-7}$. Generalizing from a vortex ring of length $2\pi R$ to a vortex tangle of length $\Lambda = LV$ and using $V_{H_2} = 4N\pi a^3/3$, the superfluid Stokes number can be estimated as

$$S = \delta\beta \approx \frac{3\phi\beta}{4a\pi}\left(\frac{\ell}{a}\right)^2, \tag{29}$$

where $\delta = Na/(2\pi R) \approx (N/V)(V/\Lambda)a = (N/V)(a/L)$. If one tenth of the tracers are trapped inside vortices, using $\beta \approx 20$ (see Fig. 5) we obtain $S \approx 5 \times 10^{-4} \ll 1$. We conclude that in this particular experiment the tracers have not disturbed the dynamics of the vortex tangle.

V. DISCUSSION

We have derived the equations of motion of a vortex ring which contains N buoyant tracer particles trapped on the vortex core. We have shown typical solutions of these equations to illustrate the difference between the motion of a bare, undisturbed vortex ring, and that of a vortex ring which is visualized by tracer particles.

A first approximate measure of the disturbance caused by trapped particles is the geometrical quantity δ given by Eq. (27), the separation between tracers along the ring in unit of tracer's size. Clearly δ must be small for tracers not to disturb the vortex. A more precise measure is the superfluid Stokes number $S = \delta\beta$, where the dimensionless temperature-dependent quantity β is approximately constant for $T < 2$ K but becomes very large for $T > 2$ K.

Finally, we have shown how this criterion is relevant to quantum turbulence and can be used to determine the maximum hydrogen-volume factor which can be used to visualize the turbulent flow without disturbing it. We have applied the criterion to a recent experiment[10] in which the drift of vortex lines generated by a heat flux was measured and found that the disturbance was negligible. It would be interesting to apply the same criterion to experiments in which vortex reconnections were detected:[12] this case is more challenging, as during the reconnection process much smaller values of the radius of curvature are expected to arise, which may be disturbed by tracer particles. Future work will attempt to make contact with a very recent experiment on the decay of vortex rings which has been performed by G. P. Bewley and K. R. Sreenivasan[20].

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